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Rings with Boolean Lattices of One-Sided Annihilators

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Abstract: The present paper is part of the research on the description of rings with a given property of the lattice of left (right) annihilators. The anti-isomorphism of lattices of left and right annihilators in any ring gives some kind of symmetry: the lattice of left annihilators is Boolean (complemented, distributive) if and only if the lattice of right annihilators is such. This allows us to restrict our investigations mainly to the left side. For a unital associative ring R, we prove that the lattice of left annihilators in R is Boolean if and only if R is a reduced ring. We also prove that the lattice of left annihilators of R being two-sided ideals is complemented if and only if this lattice is Boolean. The last statement, in turn, is known to be equivalent to the semiprimeness of R. On the other hand, for any complete lattice L, we construct a nilpotent ring whose lattice of left annihilators coincides with its sublattice of left annihilators being two-sided ideals and is isomorphic to L. This construction shows that the assumption of R being unital cannot be dropped in any of the above two results. Some additional results on rings with distributive or complemented lattices of left annihilators are obtained.

Keywords: distributive lattice; complemented lattice; Boolean lattice; annihilator in ring; reduced ring; semiprime ring

MSC: 2020: 06B15; 16N60; 16P60

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1. Introduction

Throughout the present paper, all rings are associative and, apart from Theorem 8, all rings contain $1 \neq 0$.

Rings in which the lattices of one-sided annihilators coincide with the lattices of one-sided ideals are well known in the literature under the name of dual rings or abbreviated as D-rings (see, e.g., [1–3]). Artinian D-rings, known as quasi-Frobenius rings (QF rings), have been extensively studied — for example, in [4,5]. In the case of D-rings, the lattices of one-sided annihilators are modular but, in general, these lattices can be quite complicated. Examples of rings with strange properties of lattices of annihilators are constructed, for example, in [6–9]. It is proven in [6] that every lattice can be a sublattice of the lattice of one-sided annihilators of a ring (even a commutative ring; see [7]). Rings in which the lattices of one-sided annihilators have a simple structure, i.e., they are chains, have also been investigated; see, e.g., [10,11]. The present paper is part of the research on the description of rings with a given property of the lattice of left annihilators. More precisely, we deal with rings for which this lattice is distributive or complemented, but we focus mainly on the case wherein this lattice is Boolean.

In this paper, we study one-sided annihilators and thus we focus on noncommutative rings. A similar study, but for commutative rings, was carried out by J. Lambek in the 1960s. He proved that the annihilator ideals in a commutative semiprime ring R form a complete Boolean lattice (see [12], p. 43). This result was then generalized by S. Steinberg in [13] for noncommutative semiprime rings. Recently, in [14], T. Dube and A. Taherifar proved for any ring R that the lattice of right annihilators of R that are two-sided ideals is Boolean if and only if R is semiprime.

Many properties of rings can be expressed in terms of annihilators, e.g., a domain is a ring in which the lattice of left (equivalently, right) annihilators is a two-element chain; a Boolean ring is a ring in which the left (equivalently, right) annihilators of distinct elements are distinct (see [15]); an Armendariz ring is a ring R whose lattice of left annihilators is isomorphic to the lattice of the left annihilators of the polynomial ring $R[x]$ (see [16]). The main result of the present article, Theorem 6, characterizes unital reduced rings as rings with Boolean lattices of one-sided annihilators.

In Section 2, we recall some definitions and properties related to lattices of one-sided ideals and one-sided annihilators, establish notation and indicate connections between these lattices. In Section 3, we first prove that if the lattice of the left annihilators of a unital ring R is Boolean, then R is reduced. Then, to show that the converse of this theorem is also true, we study lattices of one-sided annihilators which are two-sided ideals, obtaining as a result the above-mentioned Theorem 6: for any unital ring R, its lattice of left annihilators is Boolean if and only if R is reduced. We also show that the lattice of left annihilators that are two-sided ideals of R is Boolean if and only if this lattice is complemented; furthermore, we provide a new proof of the known fact (see [14]) that this lattice is Boolean if and only if R is semiprime. In each of the aforementioned theorems, the assumption of the existence a unity in R is essential. It is an immediate consequence of Theorem 8, in which we prove that, for any complete lattice L, there exists a nilpotent ring whose lattice of left annihilators coincides with its sublattice of left annihilators being two-sided ideals and is isomorphic to L.

We adopt the ring-theoretic terminology from [4,17]. Some facts are justified using particular rings, which are mostly semigroup algebras and monoid algebras. A description of the properties of such algebras can be found, e.g., in [18,19], but we focus only on their annihilator properties. Lattice-theoretic terminology and notation will be standard, pursuant to the usage in [20]. All considered lattices are bounded with the least element ω and the greatest element Ω.

2. Preliminaries

Let R be any unital ring. We adopt the following notation:

$\mathcal{I}_l(R)$—the set of all left ideals of R,

$\mathcal{I}_r(R)$—the set of all right ideals of R,

$\mathcal{I}(R)$—the set of all two-sided ideals of R.

The fact that I is a two-sided ideal of R we denote by $I \lhd R$. It will cause no confusion if we write an *ideal* instead of a *two-sided ideal*.

For every ring R, the sets $\mathcal{I}_l(R), \mathcal{I}_r(R), \mathcal{I}(R)$ ordered by inclusion are lattices with operations:

$$I \vee J = I + J \quad \text{and} \quad I \wedge J = I \cap J. \tag{1}$$

All these lattices have $\omega = 0$, $\Omega = R$ and are complete and modular. Moreover,

$$\mathcal{I}(R) = \mathcal{I}_l(R) \cap \mathcal{I}_r(R)$$

Hence, $\mathcal{I}(R)$ is a sublattice of $\mathcal{I}_l(R)$ and a sublattice of $\mathcal{I}_r(R)$. Obviously, if R is commutative, then $\mathcal{I}_l(R) = \mathcal{I}_r(R) = \mathcal{I}(R)$.

We are interested in rings in which the lattices of one-sided ideals are distributive or complemented. From [17] (Exercise 7 to §1), it follows that a left ideal I of a ring R has a complement in the lattice $\mathcal{I}_l(R)$ if and only if $I = Re$ for some idempotent $e \in R$, in which case $R(1-e)$ is a complement of I. An analogous result holds for the lattice $\mathcal{I}_r(R)$. A two-sided ideal $I \lhd R$ has a complement in $\mathcal{I}(R)$ if and only if $I = Re$ for some central idempotent $e \in R$. Moreover, central idempotents in a ring R form a Boolean lattice (see [4], Exercise 30 to §7). According to (3.5) in [17], we can notice the following.

Proposition 1. *For every ring R, the following statements are equivalent:*

(a) *The lattice* $\mathcal{I}_l(R)$ *is complemented;*

(b) *The lattice $\mathcal{I}_r(R)$ is complemented;*
(c) *R is isomorphic to a direct sum of finitely many matrix rings $M_{n_i}(D_i)$ over division rings D_i, for some positive integers n_i.*

The distributivity of the lattice $\mathcal{I}_l(R)$ and the distributivity of the lattice $\mathcal{I}_r(R)$ are independent properties — an example of a ring R in which $\mathcal{I}_l(R)$ is distributive (a chain of three elements) but $\mathcal{I}_r(R)$ is not distributive is given in [6]. Note that for any matrix ring $A = M_n(R)$ with $n > 1$, the lattices $\mathcal{I}_l(A)$ and $\mathcal{I}_r(A)$ are not distributive.

The following two known results characterize rings R for which the lattice $\mathcal{I}_r(R)$, resp. $\mathcal{I}(R)$, is Boolean.

Theorem 1 ([21])**.** *For every ring R, the following statements are equivalent:*

(a) *The lattice $\mathcal{I}_r(R)$ is Boolean;*
(b) *R is isomorphic to a finite direct sum of division rings.*

Theorem 2 ([21])**.** *For every ring R, the following statements are equivalent:*

(a) *The lattice $\mathcal{I}(R)$ is Boolean;*
(b) *The lattice $\mathcal{I}(R)$ is complemented;*
(c) *R is isomorphic to a finite direct sum of simple rings.*

If X is a subset of a ring R, then let $\mathrm{l}(X)$ be the left annihilator of X in R and let $\mathrm{r}(X)$ be the right annihilator of X in R, where $\mathrm{l}(X) = \{r \in R : rX = 0\}$ and $\mathrm{r}(X) = \{r \in R : Xr = 0\}$. Thus, every left annihilator is a left ideal, and every right annihilator is a right ideal in R. If $X = \{x\}$ is a one-element set, then the left and the right annihilators of X will be denoted shortly by $\mathrm{l}(x)$ and $\mathrm{r}(x)$, respectively.

The following properties of annihilators follow directly from the above definitions.

Lemma 1. *Ler R be a ring. Then:*

(1) *For any $X \subseteq R$, the set $\mathrm{l}(\mathrm{r}(X))$ is the smallest left annihilator containing X.*
(2) *If $I \lhd R$, then also $\mathrm{l}(I) \lhd R$ and $\mathrm{r}(I) \lhd R$.*
(3) *If $I \lhd R$, then also $\mathrm{l}(\mathrm{r}(I)) \lhd R$.*

Let $\mathcal{A}_l(R)$ be the set of all left annihilators in R and let $\mathcal{A}_r(R)$ be the set of all right annihilators in R. Then, $\mathcal{A}_l(R)$ is a complete lattice with operations:

$$\bigvee_{s\in S} J_s = \mathrm{l}(\mathrm{r}(\sum_{s\in S} J_s)) \quad \text{and} \quad \bigwedge_{s\in S} J_s = \bigcap_{s\in S} J_s,$$

for every family $\{J_s\}_{s\in S} \subseteq \mathcal{A}_l(R)$. Similarly, $\mathcal{A}_r(R)$ is a complete lattice with operations:

$$\bigvee_{s\in S} J_s = \mathrm{r}(\mathrm{l}(\sum_{s\in S} J_s)) \quad \text{and} \quad \bigwedge_{s\in S} J_s = \bigcap_{s\in S} J_s,$$

for every family $\{J_s\}_{s\in S} \subseteq \mathcal{A}_r(R)$. In every ring R, the lattices $\mathcal{A}_l(R)$ and $\mathcal{A}_r(R)$ have $\omega = 0$ and $\Omega = R$.

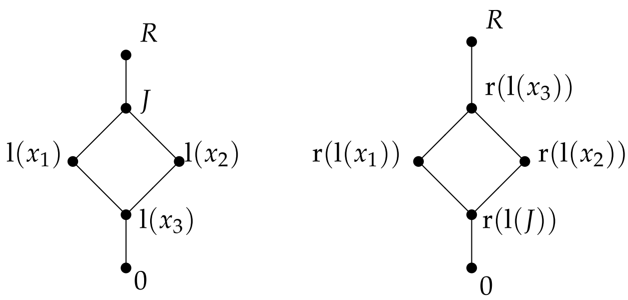
Lattices $\mathcal{A}_l(R)$ and $\mathcal{A}_r(R)$ are mutually dual. Namely, for any ring R, the mapping $\mathcal{A}_l(R) \mapsto \mathcal{A}_r(R)$ given by $I \mapsto \mathrm{r}(I)$ is an anti-isomorphism of complete lattices. The inverse function is given by $J \mapsto \mathrm{l}(J)$ for any $J \in \mathcal{A}_r(R)$. This is the Galois correspondence. Hence, $\mathcal{A}_l(R)$ is distributive (complemented, Boolean) if and only if $\mathcal{A}_r(R)$ is such. This symmetry and possible replacement of rings by their opposites allows us to restrict our investigations mainly to the lattices of left annihilators in rings. Analogous theorems for lattices of right annihilators follow directly from the above arguments.

3. Results

The lattice of left annihilators in a ring R need not be a sublattice of the lattice $\mathcal{I}_l(R)$ (see [6]) even if R is commutative (see [7]). It is natural to ask the following question: If the lattice $\mathcal{A}_l(R)$ is a sublattice of $\mathcal{I}_l(R)$, does the lattice $\mathcal{A}_r(R)$ have to be a sublattice of $\mathcal{I}_r(R)$? The answer is negative, as the following example shows.

Example 1. *Let $\mathbb{K}$ be a field and $\mathbb{K}\{x_1, x_2, x_3\}$ the polynomial ring in noncommuting variables x_1, x_2, x_3. Let I be the ideal generated by all products abc with $a, b, c \in \{x_1, x_2, x_3\}$ and by the set $\{x_1^2, x_2^2, x_3^2, x_3x_1, x_3x_2\}$. Now, let $R = \mathbb{K}\{x_1, x_2, x_3\}/I$ and let D be the ideal in R generated by all products ab with $a, b \in \{x_1, x_2, x_3\}$. Then, R is a local ring with the Jacobson radical $J = (x_1, x_2, x_3)$ (the ideal generated by x_1, x_2, x_3) and the ideal D is contained in every nonzero left annihilator. Moreover, $\mathrm{l}(D) = J$ and $\mathrm{l}(J) = \mathbb{K}x_3 + D$. To find all the remaining nontrivial left annihilators in R, it is enough to consider left annihilators of sets S such that $D \subset S \subset J$. Let $r \notin D$ and $r = \sum_i \alpha_i x_i + d$, where $\alpha_i \in \mathbb{K}$ and $d \in D$. Then, $\mathrm{l}(r) = \cap_i \mathrm{l}(x_i)$ where $\alpha_i \neq 0$. Hence, for any set S, only left annihilators of elements x_1, x_2, x_3 and their intersections are important.*

It is easy to check that $\mathrm{l}(x_1) = \mathbb{K}x_1 + \mathbb{K}x_3 + D, \mathrm{l}(x_2) = \mathbb{K}x_2 + \mathbb{K}x_3 + D, \mathrm{l}(x_3) = \mathbb{K}x_3 + D$. This means that $\mathcal{A}_l(R)$ and $\mathcal{A}_r(R)$ are as follows:



where:
$\mathrm{r}(\mathrm{l}(J)) = D$
$\mathrm{r}(\mathrm{l}(x_1)) = \mathbb{K}x_1 + D$
$\mathrm{r}(\mathrm{l}(x_2)) = \mathbb{K}x_2 + D$
$\mathrm{r}(\mathrm{l}(x_3)) = J$

It is easily seen that $\mathcal{A}_l(R)$ is a sublattice of $\mathcal{I}_l(R)$ whereas $\mathcal{A}_r(R)$ is not a sublattice of $\mathcal{I}_r(R)$.

Below, we present a well-known result of a strictly lattice-theoretic nature.

Lemma 2 ([20], I.6)**.** *Let L be a bounded distributive lattice with the smallest element ω. Then,*

(1) *Every element in L has, at most, one complement.*
(2) *If $a, b, b' \in L$, where b' is a complement of b in L, then $a \wedge b = \omega \Leftrightarrow a \leqslant b'$.*

Now, we prove one of the main results of the paper.

Theorem 3. *Let R be a ring. If the lattice $\mathcal{A}_l(R)$ is Boolean, then R is reduced.*

Proof. Assume that the lattice $\mathcal{A}_l(R)$ is Boolean and let $0 \neq n \in R$ be any element with $n^2 = 0$. Let $S = \mathrm{l}(\mathrm{r}(S))$ be a complement of $\mathrm{l}(n)$ in $\mathcal{A}_l(R)$. Since $n^2 = 0$, we have $\mathrm{l}(n) \neq 0$ and $0 \neq S \neq R$. Throughout the proof, T denotes $\mathrm{r}(S)$. Since $st = 0$ for any $s \in S$ and $t \in T$, it follows immediately that

$$\forall s \in S\ \forall t \in T\ (sn + s)(t - nt) = 0. \tag{2}$$

Let

$$W = \{sn + s : s \in S\}.$$

We claim that

$$W \neq 0 \text{ and } W \not\subseteq \mathrm{l}(n). \tag{3}$$

Indeed, if $sn + s = 0$ for any $s \in S$, then a multiplication on the right by n gives $sn = 0$. However, this contradicts the fact that $S \cap \mathrm{l}(n) = 0$ as S is a complement of $\mathrm{l}(n)$. Therefore, $W \neq 0$. Similarly, a contradiction is obtained if $(sn + s)n = 0$ for any $s \in S$. Hence, $W \not\subseteq \mathrm{l}(n)$.

Let

$$W_1 = \mathrm{l}(\mathrm{r}(W)).$$

The set W_1 is nonzero as $0 \neq W \subseteq W_1$. Our next claim is that

$$W_1 \cap S = 0. \tag{4}$$

Let $x \in W_1 \cap S$. Then, $x\mathrm{r}(W) = 0$. In particular, by (2), for every $t \in T$, $x(t - nt) = 0$. Since $x \in S$, then $xt = 0$, and we deduce that $xnt = 0$ for every $t \in T$. Consequently, $xn \in \mathrm{l}(T) = \mathrm{l}(\mathrm{r}(S)) = S$ and of course $xn \in \mathrm{l}(n)$. Since $xn \in S \cap \mathrm{l}(n) = 0$, then $xn = 0$ and finally $x \in S \cap \mathrm{l}(n) = 0$, which proves (4).

Now, by $W \subseteq W_1$ and (4), we have $W \cap S = 0$. From this and Lemma 2(2), it follows that $W \subseteq \mathrm{l}(n)$. Taking into account (3), we obtain a contradiction. Thus, the assumption that R has a nilpotent element was incorrect, which completes the proof of the theorem. □

Recall that if e is an idempotent in a ring R, then Re is the left annihilator of the element $1 - e$. We have a natural embedding of the Boolean lattice of central idempotents of R into $\mathcal{A}_l(R)$ given by $e \mapsto Re$.

Theorem 4. *Let R be a ring such that $\mathcal{A}_l(R)$ is distributive. Then, each idempotent $e \in R$ is central.*

Proof. Let $e, r \in R$ and $e = e^2$. Let $f = e + er(1 - e)$. Then, f is an idempotent in R such that $ef = f$ and $fe = e$. It is easy to check that $R = Rf \oplus R(1 - e)$ as a direct sum of subgroups. Moreover, if $e \neq f$, then a routine check shows that $Re \neq Rf$.

Let us assume now that e is not a central idempotent. Then, for a certain element $a \in R$, we have $ea(1 - e) \neq 0$ or $(1 - e)ae \neq 0$. Let $ea(1 - e) \neq 0$. Then, the element $f = e + ea(1 - e)$ is a nonzero idempotent not equal to e. Hence, Rf and Re are different complements of $R(1 - e)$ in $\mathcal{A}_l(R)$. This is impossible in a distributive lattice.

The same reasoning applies to $(1 - e)ae \neq 0$ and $g = (1 - e) + (1 - e)ae$. In this case, $g \neq 1 - e$ is an idempotent with $R = Rg \oplus Re$. □

Proposition 2. *Let R be a ring such that $\mathcal{A}_l(R)$ is complemented. If $0 \neq I$ is a nilpotent ideal in R, then I is not a left annihilator in R.*

Proof. Let $0 \neq I \lhd R, I^n = 0$ and $I \in \mathcal{A}_l(R)$. Let J be a complement of I in $\mathcal{A}_l(R)$. By Lemma 1(2), $\mathrm{r}(I) \lhd R$ and then $\mathrm{r}(J)\mathrm{r}(I) \subseteq \mathrm{r}(J) \cap \mathrm{r}(I) = 0$. This yields $\mathrm{r}(J) \subseteq \mathrm{l}(\mathrm{r}(I)) = I$ and thus $I^{n-1}\mathrm{r}(J) \subseteq I^n = 0$. Hence, $I^{n-1} \subseteq \mathrm{l}(\mathrm{r}(J)) = J$ and $I^{n-1} \subseteq I \cap J = 0$ follows. By induction, we obtain $I = 0$, as desired. □

Corollary 1. *Let R be a ring such that $\mathcal{A}_l(R)$ is complemented. If $0 \neq I$ is a nilpotent left ideal of R, then I is not a coatom in $\mathcal{A}_l(R)$.*

Proof. Let $I^n = 0$ and I be a coatom in $\mathcal{A}_l(R)$. Then, $\mathrm{r}(I)$ is an atom in $\mathcal{A}_r(R)$. Since $(IR)^n = 0$, then $\mathrm{r}(IR) \neq 0$. From this and $I \subseteq IR$, it follows that $\mathrm{r}(I) = \mathrm{r}(IR) \lhd R$. Hence, $\mathrm{l}(\mathrm{r}(I)) = I \lhd R$, a contradiction with Proposition 2. □

For any ring R, we can consider the set of those one-sided annihilators that are two-sided ideals. Let I, J be left annihilators in R that are two-sided ideals. Then, their meet $I \wedge J = I \cap J$ is a two-sided ideal and a left annihilator. Similarly, by Lemma 1, $I \vee J = \mathrm{l}(\mathrm{r}(I) \cap \mathrm{r}(J))$ is a two-sided ideal and a left annihilator. Thus, the set of all two-sided ideals that are left annihilators forms a sublattice of $\mathcal{A}_l(R)$ with the same meet and join operations. This sublattice is denoted by $\mathcal{B}_l(R)$. Analogously, the set $\mathcal{B}_r(R)$ of all two-sided ideals of R that are right annihilators is a sublattice of $\mathcal{A}_r(R)$.

The Galois correspondence $\mathcal{A}_l(R) \to \mathcal{A}_r(R)$ given by $I \to \mathrm{r}(I)$ can be restricted to the lattices $\mathcal{B}_l(R)$ and $\mathcal{B}_r(R)$. Due to this duality of lattices, we can restrict our consideration to the lattice $\mathcal{B}_l(R)$.

If R is a ring in which every left annihilator $I = \mathrm{l}(S_1)$ is also a right annihilator $I = \mathrm{r}(S_2)$ and, at the same time, every right annihilator $J = \mathrm{r}(T_1)$ is also a left annihilator $J = \mathrm{l}(T_2)$ for some subsets S_1, S_2, T_1, T_2 of R, then we write $\mathcal{A}_l(R) = \mathcal{A}_r(R)$. Analogously, if this property is satisfied for all one-sided annihilators that are two-sided ideals, then we write $\mathcal{B}_l(R) = \mathcal{B}_r(R)$.

Directly from the above notation, we obtain the following observation.

Proposition 3. *Let R be a ring. Then,*

(1) *if R is commutative, then* $\mathcal{A}_l(R) = \mathcal{A}_r(R)$;
(2) *if* $\mathcal{A}_l(R) = \mathcal{A}_r(R)$, *then* $\mathcal{A}_l(R) = \mathcal{A}_r(R) = \mathcal{B}_l(R) = \mathcal{B}_r(R)$;
(3) *if* $\mathcal{A}_l(R) = \mathcal{B}_l(R)$, *then* $\mathcal{A}_r(R) = \mathcal{B}_r(R)$.

Below, we have a simple example of a ring R in which $\mathcal{B}_l(R) = \mathcal{B}_r(R)$ is a Boolean lattice and this does not imply that $\mathcal{A}_l(R)$ is Boolean.

Example 2. *Let* $R = M_2(\mathbb{K})$ *be the full ring of 2-by-2 matrices over a field* $\mathbb{K}$. *Then, R is a simple ring and* $\mathcal{B}_l(R) = \mathcal{B}_r(R)$ *is a Boolean lattice with two elements* $0, R$. *Moreover, the lattices* $\mathcal{A}_l(R)$ *and* $\mathcal{A}_r(R)$ *are not distributive.*

A ring R is said to be semicommutative if $ab = 0$ implies $aRb = 0$ for all $a, b \in R$. The lattice notation, introduced above, allows us to say that the ring R is semicommutative if and only if $\mathcal{A}_l(R) = \mathcal{B}_l(R)$ (and then by Proposition 3(3) also $\mathcal{A}_r(R) = \mathcal{B}_r(R)$). A ring R is called reversible if $xy = 0$ implies $yx = 0$ for all $x, y \in R$. Reversible rings are examples of rings R with the property $\mathcal{A}_l(R) = \mathcal{A}_r(R)$. Analogously, examples of rings R satisfying the property $\mathcal{B}_l(R) = \mathcal{B}_r(R)$ are rings known in the literature as reflexive rings. Recall that a ring R is called reflexive if $IJ = 0$ implies $JI = 0$ for all ideals I, J of R. However, lattices of annihilators do not characterize either reversible or reflexive rings. Below, we construct an example of a ring R with $\mathcal{A}_l(R) = \mathcal{A}_r(R)$ that is neither reversible nor reflexive.

Example 3. *Let* $\mathbb{K}$ *be a field and* $A = \mathbb{K}\{x, y, z\}$ *be the polynomial ring in noncommuting variables* x, y, z. *Let I be the ideal of A generated by the sets* $\{xy, yz, zx\}$ *and* $\{abc : a, b, c \in \{x, y, z\}\}$. *Now, let* $R = A/I$. *Then, R is a local ring with the Jacobson radical* $J = (x, y, z)$.

We first prove that $\mathcal{A}_l(R) = \mathcal{A}_r(R)$. *Clearly,* $D = J^2 = \{ab : a, b \in \{x, y, z\}\}$ *is contained in every left and right nonzero annihilator. More precisely, D is an atom in the lattices* $\mathcal{A}_l(R)$ *and* $\mathcal{A}_r(R)$, *while J is a coatom. An easy computation shows that*

$$\mathrm{l}(x) = \mathrm{r}(y) = \mathbb{K}z + D, \mathrm{l}(y) = \mathrm{r}(z) = \mathbb{K}x + D, \mathrm{l}(z) = \mathrm{r}(x) = \mathbb{K}y + D. \quad (5)$$

It is not difficult to verify (by a similar method as in Example 1) that we have received all nontrivial left and right annihilators in R. Hence, the lattices $\mathcal{A}_l(R)$ *and* $\mathcal{A}_r(R)$ *are of the shape given in Figure 1.*

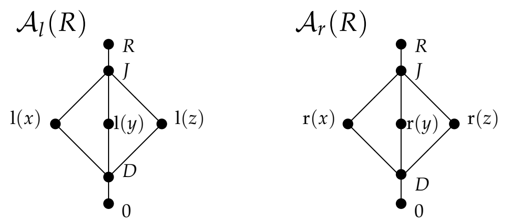


Figure 1. Lattices $\mathcal{A}_l(R)$ and $\mathcal{A}_l(R)$ for the ring R given in Example 3.

By (5), we have $\mathcal{A}_l(R) = \mathcal{A}_r(R)$. Now, by Proposition 3(2), $\mathcal{B}_l(R) = \mathcal{B}_r(R)$. As $xy = 0$ and $yx \neq 0$, the ring R is not reversible. Since, for the ideals $I_1 = \mathbb{K}x + D$, $I_2 = \mathbb{K}y + D$ of R, we have $I_1 I_2 = 0$ and $I_2 I_1 \neq 0$, the ring R is not reflexive either.

A very important subclass of reflexive rings is the class of semiprime rings. Recall that, in a semiprime ring R, for each ideal I of R, we have $\mathrm{l}(I) = \mathrm{r}(I)$. Furthermore, we have the following known result.

Theorem 5 (See [13], Exercise 1.2.7)**.** *Let R be a semiprime ring. Then, $\mathcal{B}_l(R) = \mathcal{B}_r(R)$ is a Boolean lattice.*

It is clear by Example 3 that the semiprime property is relevant in the assumption of the above theorem.

We now turn our attention to the reduced rings. If R is a reduced ring, then R is semiprime, and for every $x \in R$, we have $\mathrm{l}(x) = \mathrm{r}(x)$. Hence, $\mathcal{A}_l(R) = \mathcal{A}_r(R)$ holds for any reduced ring R. From this, Proposition 3(2) and the above Theorem 5, we obtain the following corollary.

Corollary 2. *If R is a reduced ring, then $\mathcal{A}_l(R) = \mathcal{A}_r(R)$ is a Boolean lattice.*

By combining Corollary 2 and Theorem 3, we obtain

Theorem 6. *For any ring R, the lattice $\mathcal{A}_l(R)$ is Boolean if and only if R is reduced.*

We can formulate the following corollary.

Corollary 3. *Let R be a ring. If the lattice $\mathcal{A}_l(R)$ is Boolean, then $\mathcal{A}_l(R) = \mathcal{A}_r(R)$ and $\mathrm{l}(S) = \mathrm{r}(S)$ for every $S \subseteq R$.*

We now return to the lattice $\mathcal{B}_l(R)$ and formulate our main result for this lattice. The equivalence of conditions (a) and (c) of the following theorem was shown in Theorem 1.1 of [14], but in our proof, we use arguments different from those in [14].

Theorem 7. *For any ring R, the following conditions are equivalent.*

(a) *The lattice $\mathcal{B}_l(R)$ is Boolean.*
(b) *The lattice $\mathcal{B}_l(R)$ is complemented.*
(c) *R is a semiprime ring.*

Proof. (a) $\Rightarrow$ (b). This implication is obvious.

(b) $\Rightarrow$ (c). Let $0 \neq I$ be a nilpotent ideal in R, and $I^n = 0$, $I^{n-1} \neq 0$. Thus, without loss of generality, we may assume that $n = 2$. Then, $\mathrm{r}(\mathrm{l}(I))$ is a nonzero ideal different from R. Since $\mathcal{B}_l(R)$ is complemented, $\mathcal{B}_r(R)$ is complemented as well. Let J be a complement of $\mathrm{r}(\mathrm{l}(I))$ in $\mathcal{B}_r(R)$. From $I \subseteq \mathrm{r}(\mathrm{l}(I))$ and $\mathrm{r}(\mathrm{l}(I)) \cap J = 0$, it follows that

$$IJ \subseteq I \cap J = 0. \tag{6}$$

Since $0 \neq \mathrm{l}(I) = \mathrm{l}(\mathrm{r}(\mathrm{l}(I)))$ is a complement of $\mathrm{l}(J)$ in $\mathcal{B}_l(R)$, then $\mathrm{l}(I) \cap \mathrm{l}(J) = 0$. From this and $I \subseteq \mathrm{l}(I)$, it follows that $I \cap \mathrm{l}(J) = 0$. This implies $IJ \neq 0$ and we have a contradiction with formula (6), which implies (c).

(c) $\Rightarrow$ (a). This implication is a direct consequence of Theorem 5. □

Since, for any semiprime ring R, we have $\mathcal{B}_l(R) = \mathcal{B}_r(R)$, the following corollary is an immediate consequence of Theorem 7.

Corollary 4. *Let R be a ring. If the lattice $\mathcal{B}_l(R)$ is complemented, then $\mathcal{B}_l(R) = \mathcal{B}_r(R)$ and $\mathrm{l}(S) = \mathrm{r}(S)$ for every $S \lhd R$.*

We now show that in Theorems 6 and 7, the assumption that R is a ring with an identity element cannot be dropped. In particular, the proof of the following theorem provides a method for constructing rings (without identity) in which the lattices of annihilators are Boolean but the rings are neither reduced nor semiprime.

Theorem 8. *For any complete lattice L (in particular, any complete Boolean lattice), there exists a ring R_L without identity such that R_L is nilpotent and L is isomorphic to $\mathcal{A}_l(R_L) = \mathcal{B}_l(R_L)$.*

Proof. We use a modification of the construction given in [6]. Let L be a complete lattice and let $\mathrm{M}(L)$ be the free semigroup with the set L of free generators. Let I be the smallest ideal of $\mathrm{M}(L)$ containing all products xyz, where $x, y, z \in L$, and all products xy, where $x, y \in L$ and $x \leq y$. Put $\overline{L} = \mathrm{M}(L)/I$, the Rees factor semigroup. Clearly, $L \subseteq \overline{L}$ in a natural way and $L^2 = \{0\} \cup \{xy : x, y \in L,\ x \not\leq y\}$. Moreover, $\overline{L} = L \cup L^2$. For us, the following *uniquenes property* in $\overline{L}$ is crucial: if $x_1, x_2, y_1, y_2 \in \overline{L}$ are such that $x_1x_2 = y_1y_2 \neq 0$, then $x_1 = y_1$ and $x_2 = y_2$.

Now, let $\mathbb{K}$ be a field and $R_L = \mathbb{K}_0[\overline{L}]$ the contracted semigroup algebra. Then, R_L is a nilpotent algebra over the field $\mathbb{K}$ (since $(R_L)^3 = 0$) and a direct sum of linear spaces

$$R_L = V \oplus V^2,$$

where the natural base of V can be identified with L and the natural base of V^2 can be identified with $L^2 \setminus \{0\}$.

Now, let $\phi : L \longrightarrow \mathcal{A}_l(R_L))$ be given by $\phi(x) = \mathrm{l}(x)$ for $x \in L$. We claim that ϕ is an isomorphism of lattices.

Observe first that, for every element $x \in L \subset V$, we have

$$V^2 \subseteq \phi(x) = \mathrm{l}(x) = V_x \oplus V^2,$$

where $V_x \subseteq V$ is the subspace spanned by all $y \in L$ with $y \leq x$. In particular, $x \in \phi(x)$ and ϕ is an order-preserving embedding.

Now, if $\{z_i\}, \{w_j\} \subseteq V$ are finite subsets such that $\sum_{ij} \alpha_{ij} z_i w_j = 0$ for some $\alpha_{ij} \in \mathbb{K}$, then, by the uniqueness property of the semigroup $\overline{L}$, $\alpha_{ij} z_i w_j = 0$ for every pair i, j. Thus, if $s = \sum_j \alpha_j w_j \in V$, where $\alpha_j \neq 0$, then $\mathrm{l}(s) = \cap_j \mathrm{l}(w_j) = \mathrm{l}(\bigwedge_j w_j)$. It turns out that for any $S \subseteq R$, the left annihilator $\mathrm{l}(S)$ is a left annihilator of an element of L, as $\mathrm{l}(S) = \bigcap_{s \in S} \mathrm{l}(s)$. Since $\mathrm{l}(x) R_L \subseteq V^2 \subseteq \mathrm{l}(x)$ for every $x \in L$, we conclude that every left annihilator in R_L is in fact a two-sided ideal. Thus, $\mathcal{A}_l(R_L) = \mathcal{B}_l(R_L)$. □

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